

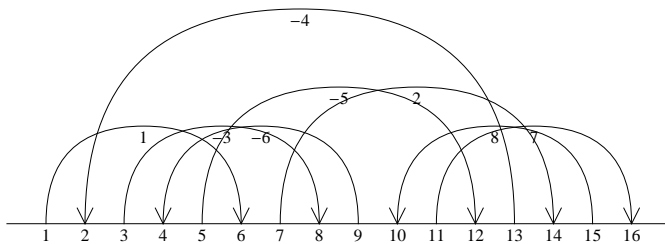
Pensieve Header: Poly-only  $\beta$ -Alexander.

```
<< KnotTheory`  
GD[K_] := GD @@ (  
  PD[K] /. X[i_, j_, k_, l_] => If[PositiveQ[X[i, j, k, l]],  
    Ar[l, i, +1], Ar[j, i, -1]  
  ]  
)
```

Loading KnotTheory` version of August 22, 2010, 13:36:57.55.  
Read more at <http://katlas.org/wiki/KnotTheory>.

```
Draw[expr_] := expr /. gd_GD => Draw[gd];  
Draw[gd_GD] := Module[  
  {n = Length[gd], h, k = 0},  
  Graphics[  
    Line[{{0, 0}, {2 n + 1, 0}}],  
    Table[Text[i, {i, -0.3}], {i, 2 n}],  
    (List @@ gd) /. {  
      Ar[i_, j_, s_] => {  
        h = Abs[i - j] / 2;  
        BezierCurve[  
          {i, 0}, {i, h}, {(i + j) / 2, h}, {j, h}, {j, 0}  
        ], SplineDegree -> 2],  
        Text[s * (++k), {(i + j) / 2, h - 0.3}],  
        Line[{{j - 0.2, 0.4}, {j, 0}, {j + 0.2, 0.4}}]  
      }  
    ]  
  ]  
];  
Draw[GD[Knot[8, 17]]]
```

KnotTheory::loading: Loading precomputed data in PD4Knots`.



```

βSimplify = Factor;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, σ_, μ_]] := B[
  βSimplify[ω],
  Collect[σ, _h, βSimplify],
  Collect[μ, _h, Collect[#, _t, βSimplify] &]
];
(* "L" for "Labels" *)
hL[β_] := Union[Cases[β, h[s_] → s, Infinity]];
tL[β_] := Union[Cases[β, t[s_] | T_s_ → s, Infinity]];
gL[β_] := Union[hL[β], tL[β]];
SetAttributes[βForm, Listable];
βForm[B[ω_, σ_, μ_]] := Module[
  {tails, heads, mat},
  tails = tL[B[ω, σ, μ]]; heads = hL[B[ω, σ, μ]];
  mat = Outer[βSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
  PrependTo[mat, t /@ tails];
  mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
  AppendTo[mat, Prepend[βSimplify[Coefficient[σ, h[#]]] & /@ heads, "σ"]];
  MatrixForm[mat]
];

R[x_, y_] := B[1, T_x h[y], (T_x - 1) t[x] h[y]];
Rinv[x_, y_] := B[1, T_x^-1 h[y], (T_x^-1 - 1) t[x] h[y]];
tm[x_, y_, z_][β_] := β /. {t[x] → t[z], t[y] → t[z], T_x → T_z, T_y → T_z};
hm[x_, y_, z_][B[ω_, σ_, μ_]] := Module[
  {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
  B[ω,
    (σ /. h[x] | h[y] → 0) +
    h[z] Replace[D[σ, h[x]], 0 → 1] Replace[D[σ, h[y]], 0 → 1],
    M + h[z] (γx + Replace[D[σ, h[x]], 0 → 1] γy)
  ] // βCollect
];

swap[x_, y_][B[ω_, σ_, μ_]] := Module[
  {α, β, γ, δ, ε},
  α = Coefficient[μ, h[x] t[y]];
  β = D[μ, t[y]] /. h[x] → 0;
  γ = D[μ, h[x]] /. t[y] → 0;
  δ = μ /. h[x] | t[y] → 0;
  ε = 1 + α;
  B[ω * ε, σ, Plus[
    α (1 + (Replace[D[σ, h[x]], 0 → 1] - 1 - α) / ε) h[x] t[y],
    β (1 + (Replace[D[σ, h[x]], 0 → 1] - 1 - α) / ε) t[y],
    γ / ε h[x],
    δ - (1 / ε) γ * β
  ]] // βCollect
];

gm[x_, y_, z_][β_] := β // swap[y, x] // hm[x, y, z] // tm[x, y, z];
B /: B[ω1_, σ1_, μ1_] B[ω2_, σ2_, μ2_] := B[ω1 * ω2, σ1 + σ2, μ1 + μ2];

```

```

{
  β = B[ω[T1, T2, T3, T4], 1, Sum[αi[T1, T2, T3, T4] t[i] h[1], {i, 4}]],
  β // tm[1, 2, 1],
  t1 = β // tm[1, 2, 1] // tm[1, 3, 1],
  t2 = β // tm[2, 3, 28] // tm[1, 28, 1],
  t1 == t2
} // βForm


$$\left( \begin{array}{cc} \omega[T_1, T_2, T_3, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_2, T_3, T_4] \\ t[2] & \alpha_2[T_1, T_2, T_3, T_4] \\ t[3] & \alpha_3[T_1, T_2, T_3, T_4] \\ t[4] & \alpha_4[T_1, T_2, T_3, T_4] \\ \sigma & 0 \end{array} \right),$$



$$\left( \begin{array}{cc} \omega[T_1, T_1, T_3, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_1, T_3, T_4] + \alpha_2[T_1, T_1, T_3, T_4] \\ t[3] & \alpha_3[T_1, T_1, T_3, T_4] \\ t[4] & \alpha_4[T_1, T_1, T_3, T_4] \\ \sigma & 0 \end{array} \right),$$



$$\left( \begin{array}{cc} \omega[T_1, T_1, T_1, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_1, T_1, T_4] + \alpha_2[T_1, T_1, T_1, T_4] + \alpha_3[T_1, T_1, T_1, T_4] \\ t[4] & \alpha_4[T_1, T_1, T_1, T_4] \\ \sigma & 0 \end{array} \right),$$



$$\left( \begin{array}{cc} \omega[T_1, T_1, T_1, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_1, T_1, T_4] + \alpha_2[T_1, T_1, T_1, T_4] + \alpha_3[T_1, T_1, T_1, T_4] \\ t[4] & \alpha_4[T_1, T_1, T_1, T_4] \\ \sigma & 0 \end{array} \right), \betaForm[True]$$


{
  β = B[ω, 1, Sum[α10i+j t[i] h[j], {i, 2}, {j, 4}]],
  β // hm[1, 2, 1],
  t1 = β // hm[1, 2, 1] // hm[1, 3, 1],
  t2 = β // hm[2, 3, 28] // hm[1, 28, 1],
  t1 == t2
} // βForm // ColumnForm


$$\left( \begin{array}{ccccc} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \sigma & 0 & 0 & 0 & 0 \end{array} \right)$$



$$\left( \begin{array}{cccc} \omega & h[1] & h[3] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \sigma & 1 & 0 & 0 \end{array} \right)$$



$$\left( \begin{array}{ccc} \omega & h[1] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} + \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} + \alpha_{23} & \alpha_{24} \\ \sigma & 1 & 0 \end{array} \right)$$



$$\left( \begin{array}{ccc} \omega & h[1] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} + \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} + \alpha_{23} & \alpha_{24} \\ \sigma & 1 & 0 \end{array} \right)$$

βForm[True]

```

```

Clear[β];
{β1 = B[ω, 1, h[1] t[1] α + h[2] t[1] β + h[1] t[2] γ + h[2] t[2] δ],
 β1 // swap[1, 1]
 } // βForm

```

$$\left\{ \begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \delta \\ \sigma & 0 & 0 \end{pmatrix}, \begin{pmatrix} (1+\alpha)\omega & h[1] & h[2] \\ t[1] & \frac{\alpha}{1+\alpha} & \frac{\beta}{1+\alpha} \\ t[2] & \frac{\gamma}{1+\alpha} & \frac{-\beta\gamma+\delta+\alpha\delta}{1+\alpha} \\ \sigma & 0 & 0 \end{pmatrix} \right\}$$

```

{
 β = B[ω, 1, Sum[α10i+j t[i] h[j], {i, 2}, {j, 3}]],
 β // hm[1, 2, 1],
 t1 = β // hm[1, 2, 1] // swap[1, 1],
 t2 = β // swap[1, 1] // swap[2, 1] // hm[1, 2, 1],
 First[t1] == First[t2],
 Last[t1] == Last[t2] // Simplify
 } // βForm // ColumnForm

```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \sigma & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \alpha_{11} + \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} + \alpha_{22} & \alpha_{23} \\ \sigma & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega (1 + \alpha_{11} + \alpha_{12}) & h[1] & h[3] \\ t[1] & \frac{\alpha_{11} + \alpha_{12}}{1 + \alpha_{11} + \alpha_{12}} & \frac{\alpha_{13}}{1 + \alpha_{11} + \alpha_{12}} \\ t[2] & \frac{\alpha_{21} + \alpha_{22}}{1 + \alpha_{11} + \alpha_{12}} & \frac{-\alpha_{13} \alpha_{21} - \alpha_{13} \alpha_{22} + \alpha_{23} + \alpha_{11} \alpha_{23} + \alpha_{12} \alpha_{23}}{1 + \alpha_{11} + \alpha_{12}} \\ \sigma & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega (1 + \alpha_{11} + \alpha_{12}) & h[1] & h[3] \\ t[1] & \frac{\alpha_{11} + \alpha_{12}}{1 + \alpha_{11} + \alpha_{12}} & \frac{\alpha_{13}}{1 + \alpha_{11} + \alpha_{12}} \\ t[2] & \frac{\alpha_{21} + \alpha_{22}}{1 + \alpha_{11} + \alpha_{12}} & \frac{-\alpha_{13} \alpha_{21} - \alpha_{13} \alpha_{22} + \alpha_{23} + \alpha_{11} \alpha_{23} + \alpha_{12} \alpha_{23}}{1 + \alpha_{11} + \alpha_{12}} \\ \sigma & 1 & 0 \end{pmatrix}$$

```
βForm[True]
```

```
βForm[True]
```

```
{
  β = B[ω, 1, Sum[α10 i+j t[i] h[j], {i, 3}, {j, 2}]],
  t1 = β // tm[1, 2, 1] // swap[1, 1],
  t2 = β // swap[1, 2] // swap[1, 1] // tm[1, 2, 1],
  First[t1] == First[t2],
  Last[t1] == Last[t2] // Simplify
} // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \\ t[3] & \alpha_{31} & \alpha_{32} \\ \sigma & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega (1 + \alpha_{11} + \alpha_{21}) & h[1] & h[2] \\ t[1] & \frac{\alpha_{11} + \alpha_{21}}{1 + \alpha_{11} + \alpha_{21}} & \frac{\alpha_{12} + \alpha_{22}}{1 + \alpha_{11} + \alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1 + \alpha_{11} + \alpha_{21}} & \frac{-\alpha_{12} \alpha_{31} - \alpha_{22} \alpha_{31} + \alpha_{32} + \alpha_{11} \alpha_{32} + \alpha_{21} \alpha_{32}}{1 + \alpha_{11} + \alpha_{21}} \\ \sigma & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega (1 + \alpha_{11} + \alpha_{21}) & h[1] & h[2] \\ t[1] & \frac{\alpha_{11} + \alpha_{21}}{1 + \alpha_{11} + \alpha_{21}} & \frac{\alpha_{12} + \alpha_{22}}{1 + \alpha_{11} + \alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1 + \alpha_{11} + \alpha_{21}} & \frac{-\alpha_{12} \alpha_{31} - \alpha_{22} \alpha_{31} + \alpha_{32} + \alpha_{11} \alpha_{32} + \alpha_{21} \alpha_{32}}{1 + \alpha_{11} + \alpha_{21}} \\ \sigma & 0 & 0 \end{pmatrix}$$

βForm[True]

βForm[True]

```
{
  β = B[ω, 1, Sum[α10 i+j t[i] h[j], {i, 4}, {j, 4}]],
  t1 = β // gm[1, 2, 1] // gm[1, 3, 1],
  t2 = β // gm[2, 3, 2] // gm[1, 2, 1],
  First[t1] == First[t2],
  Last[t1] == Last[t2] // Simplify
} // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ t[4] & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \sigma & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega (1 + \alpha_{12} + \alpha_{13} - \alpha_{13} \alpha_{22} + \alpha_{23} + \alpha_{12} \alpha_{23}) & & & & \\ t[1] & \frac{\alpha_{11} + \alpha_{12} + \alpha_{13} + \alpha_{21} + \alpha_{12} \alpha_{21} + \alpha_{22} - \alpha_{11} \alpha_{22} - \alpha_{13} \alpha_{22} + \alpha_{23} + \alpha_{12} \alpha_{23} + \alpha_{31} + \alpha_{12} \alpha_{31} + \alpha_{13} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} +}{1 + c} & & & \\ t[4] & \frac{\alpha_{41} + \alpha_{12} \alpha_{41} + \alpha_{13} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} + \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{42} - \alpha_{11} \alpha_{42}}{1 + c} & & & \\ \sigma & & & & \end{pmatrix}$$

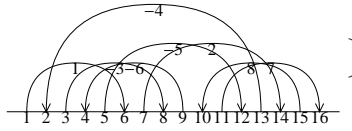
$$\begin{pmatrix} \omega (1 + \alpha_{12} + \alpha_{13} - \alpha_{13} \alpha_{22} + \alpha_{23} + \alpha_{12} \alpha_{23}) & & & & \\ t[1] & \frac{\alpha_{11} + \alpha_{12} + \alpha_{13} + \alpha_{21} + \alpha_{12} \alpha_{21} + \alpha_{22} - \alpha_{11} \alpha_{22} - \alpha_{13} \alpha_{22} + \alpha_{23} + \alpha_{12} \alpha_{23} + \alpha_{31} + \alpha_{12} \alpha_{31} + \alpha_{13} \alpha_{31} - \alpha_{13} \alpha_{22} \alpha_{31} +}{1 + c} & & & \\ t[4] & \frac{\alpha_{41} + \alpha_{12} \alpha_{41} + \alpha_{13} \alpha_{41} - \alpha_{13} \alpha_{22} \alpha_{41} + \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{12} \alpha_{23} \alpha_{41} + \alpha_{42} - \alpha_{11} \alpha_{42}}{1 + c} & & & \\ \sigma & & & & \end{pmatrix}$$

βForm[True]

βForm[True]

```
{Alexander[K = Knot[8, 17]][X], Draw[GD[K]]}
```

$$\left\{ 11 - \frac{1}{X^3} + \frac{4}{X^2} - \frac{8}{X} - 8X + 4X^2 - X^3, \right.$$



```
(β = Times @@ GD[K] /. {Ar[x_, y_, +1] => R[x, y], Ar[x_, y_, -1] => Rinv[x, y]}) //
βForm
```

	1	h[2]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	0	-1 + T <sub>1</sub>	0	0	0	0	0	0
t[3]	0	0	0	$-\frac{-1+T_3}{T_3}$	0	0	0	0	0
t[5]	0	0	0	0	0	$-\frac{-1+T_5}{T_5}$	0	0	0
t[7]	0	0	0	0	0	0	-1 + T <sub>7</sub>	0	0
t[9]	0	$-\frac{-1+T_9}{T_9}$	0	0	0	0	0	0	0
t[11]	0	0	0	0	0	0	0	-1 + T <sub>11</sub>	0
t[13]	$-\frac{-1+T_{13}}{T_{13}}$	0	0	0	0	0	0	0	0
t[15]	0	0	0	0	-1 + T <sub>15</sub>	0	0	0	0
σ	$\frac{1}{T_{13}}$	$\frac{1}{T_9}$	T <sub>1</sub>	$\frac{1}{T_3}$	T <sub>15</sub>	$\frac{1}{T_5}$	T <sub>7</sub>	T <sub>11</sub>	

```
β = Times @@ GD[K] /. {Ar[x_, y_, +1] => R[x, y], Ar[x_, y_, -1] => Rinv[x, y]};
```

```
Table[
```

```
{k, (β = β // gm[1, k, 1]) // βForm, Collect[Last[β] /. t[i_] => 1, _h, Simplify]},
```

```
{k, 2, 2 Crossings[K]}
```

```
] // ColumnForm
```

{2,	1	h[1]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]	, h[6] (-1 + T <sub>1</sub> )
	t[1]	0	0	$\frac{-1+T_1}{T_{13}}$	0	0	0	0	0	
	t[3]	0	0	0	$-\frac{-1+T_3}{T_3}$	0	0	0	0	
	t[5]	0	0	0	0	0	$-\frac{-1+T_5}{T_5}$	0	0	
	t[7]	0	0	0	0	0	0	-1 + T <sub>7</sub>	0	
	t[9]	0	$-\frac{-1+T_9}{T_9}$	0	0	0	0	0	0	
	t[11]	0	0	0	0	0	0	0	-1 + T <sub>11</sub>	
	t[13]	$-\frac{-1+T_{13}}{T_{13}}$	0	$\frac{(-1+T_1)(-1+T_{13})}{T_{13}}$	0	0	0	0	0	
t[15]	0	0	0	0	-1 + T <sub>15</sub>	0	0	0		
σ	$\frac{1}{T_{13}}$	$\frac{1}{T_9}$	T <sub>1</sub>	$\frac{1}{T_3}$	T <sub>15</sub>	$\frac{1}{T_5}$	T <sub>7</sub>	T <sub>11</sub>		
{3,	1	h[1]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]	, h[8] (-1 + $\frac{1}{T_1}$ )
	t[1]	0	0	$\frac{-1+T_1}{T_{13}}$	$-\frac{-1+T_1}{T_1}$	0	0	0	0	
	t[5]	0	0	0	0	0	$-\frac{-1+T_5}{T_5}$	0	0	
	t[7]	0	0	0	0	0	0	-1 + T <sub>7</sub>	0	
	t[9]	0	$-\frac{-1+T_9}{T_9}$	0	0	0	0	0	0	
	t[11]	0	0	0	0	0	0	0	-1 + T <sub>11</sub>	
	t[13]	$-\frac{-1+T_{13}}{T_{13}}$	0	$\frac{(-1+T_1)(-1+T_{13})}{T_{13}}$	0	0	0	0	0	
	t[15]	0	0	0	0	-1 + T <sub>15</sub>	0	0	0	
σ	$\frac{1}{T_{13}}$	$\frac{1}{T_9}$	T <sub>1</sub>	$\frac{1}{T_1}$	T <sub>15</sub>	$\frac{1}{T_5}$	T <sub>7</sub>	T <sub>11</sub>		

$$\left\{ 4, \begin{array}{c} \left( \begin{array}{cccccccc} 1 & h[1] & h[6] & h[8] & h[10] & h[12] & h[14] & h[16] \\ t[1] & 0 & \frac{-1+T_1}{T_9 T_{13}} & -\frac{-1+T_1}{T_1 T_9} & 0 & 0 & 0 & 0 \\ t[5] & 0 & 0 & 0 & 0 & -\frac{-1+T_9}{T_5} & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & 0 & -1+T_7 & 0 \\ t[9] & -\frac{-1+T_9}{T_9 T_{13}} & \frac{(-1+T_1)(-1+T_9)}{T_9 T_{13}} & -\frac{(-1+T_1)(-1+T_9)}{T_1 T_9} & 0 & 0 & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{11} \\ t[13] & -\frac{-1+T_{13}}{T_{13}} & \frac{(-1+T_1)(-1+T_{13})}{T_{13}} & 0 & 0 & 0 & 0 & 0 \\ t[15] & 0 & 0 & 0 & -1+T_{15} & 0 & 0 & 0 \\ \sigma & \frac{1}{T_9 T_{13}} & T_1 & \frac{1}{T_1} & T_{15} & \frac{1}{T_5} & T_7 & T_{11} \end{array} \right) \end{array} \right., h[8] \left( -1 + \frac{1}{T_1} \right)$$

$$\left\{ 5, \begin{array}{c} \left( \begin{array}{cccccccc} 1 & h[1] & h[6] & h[8] & h[10] & h[12] & h[14] & h[16] \\ t[1] & 0 & \frac{-1+T_1}{T_9 T_{13}} & -\frac{-1+T_1}{T_1 T_9} & 0 & -\frac{-1+T_1}{T_1} & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & 0 & -1+T_7 & 0 \\ t[9] & -\frac{-1+T_9}{T_9 T_{13}} & \frac{(-1+T_1)(-1+T_9)}{T_9 T_{13}} & -\frac{(-1+T_1)(-1+T_9)}{T_1 T_9} & 0 & 0 & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{11} \\ t[13] & -\frac{-1+T_{13}}{T_{13}} & \frac{(-1+T_1)(-1+T_{13})}{T_{13}} & 0 & 0 & 0 & 0 & 0 \\ t[15] & 0 & 0 & 0 & -1+T_{15} & 0 & 0 & 0 \\ \sigma & \frac{1}{T_9 T_{13}} & T_1 & \frac{1}{T_1} & T_{15} & \frac{1}{T_1} & T_7 & T_{11} \end{array} \right) \end{array} \right., h[8] \left( -1 + \frac{1}{T_1} \right)$$

$$\left\{ 6, \begin{array}{c} \left( \begin{array}{cccccccc} \frac{-1+T_1+T_9 T_{13}}{T_9 T_{13}} & h[1] & h[8] & h[10] & h[12] & h[14] & h[16] \\ t[1] & \frac{(-1+T_1) T_1}{T_9 T_{13} (-1+T_1+T_9 T_{13})} & -\frac{(-1+T_1) T_{13}}{-1+T_1+T_9 T_{13}} & 0 & -\frac{(-1+T_1) T_9 T_{13}}{-1+T_1+T_9 T_{13}} & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & -1+T_7 & 0 \\ t[9] & -\frac{-1+T_9}{-1+T_1+T_9 T_{13}} & -\frac{(-1+T_1)(-1+T_9) T_{13}}{T_1 (-1+T_1+T_9 T_{13})} & 0 & \frac{(-1+T_1)^2 (-1+T_9)}{T_1 (-1+T_1+T_9 T_{13})} & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & -1+T_{11} \\ t[13] & -\frac{T_9 (-1+T_{13})}{-1+T_1+T_9 T_{13}} & \frac{(-1+T_1)^2 (-1+T_{13})}{T_1 (-1+T_1+T_9 T_{13})} & 0 & \frac{(-1+T_1)^2 T_9 (-1+T_{13})}{T_1 (-1+T_1+T_9 T_{13})} & 0 & 0 \\ t[15] & 0 & 0 & -1+T_{15} & 0 & 0 & 0 \\ \sigma & \frac{T_1}{T_9 T_{13}} & \frac{1}{T_1} & T_{15} & \frac{1}{T_1} & T_7 & T_{11} \end{array} \right) \end{array} \right., h[8]$$

$$\left\{ 7, \begin{array}{c} \left( \begin{array}{cccccccc} \frac{-1+T_1+T_9 T_{13}}{T_9 T_{13}} & h[1] & h[8] & h[10] & h[12] & h[14] & h[16] \\ t[1] & \frac{(-1+T_1) T_1}{T_9 T_{13} (-1+T_1+T_9 T_{13})} & -\frac{(-1+T_1) T_{13}}{-1+T_1+T_9 T_{13}} & 0 & -\frac{(-1+T_1) T_9 T_{13}}{-1+T_1+T_9 T_{13}} & -1+T_1 & 0 \\ t[9] & -\frac{-1+T_9}{-1+T_1+T_9 T_{13}} & -\frac{(-1+T_1)(-1+T_9) T_{13}}{T_1 (-1+T_1+T_9 T_{13})} & 0 & \frac{(-1+T_1)^2 (-1+T_9)}{T_1 (-1+T_1+T_9 T_{13})} & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & -1+T_{11} \\ t[13] & -\frac{T_9 (-1+T_{13})}{-1+T_1+T_9 T_{13}} & \frac{(-1+T_1)^2 (-1+T_{13})}{T_1 (-1+T_1+T_9 T_{13})} & 0 & \frac{(-1+T_1)^2 T_9 (-1+T_{13})}{T_1 (-1+T_1+T_9 T_{13})} & 0 & 0 \\ t[15] & 0 & 0 & -1+T_{15} & 0 & 0 & 0 \\ \sigma & \frac{T_1}{T_9 T_{13}} & \frac{1}{T_1} & T_{15} & \frac{1}{T_1} & T_1 & T_{11} \end{array} \right) \end{array} \right., h[8]$$

$$\left\{ 8, \begin{array}{c} \left( \begin{array}{cccccccc} \frac{-1+T_1+T_{13}-T_1 T_{13}+T_9 T_{13}}{T_9 T_{13}} & h[1] & h[10] & h[12] & h[14] \\ t[1] & \frac{(-1+T_1)(-1+T_{13})}{T_9 T_{13} (1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13})} & 0 & \frac{(-1+T_1) T_9 T_{13}}{T_1 (1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13})} & -\frac{(-1+T_1)(-1+T_1+T_9 T_{13})}{T_1 (1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13})} \\ t[9] & -\frac{-1+T_9}{-1+T_1+T_{13}-T_1 T_{13}+T_9 T_{13}} & 0 & \frac{(-1+T_1)^2 (-1+T_9)(-1+T_{13})}{T_1 (1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13})} & -\frac{(-1+T_1)^2 (-1+T_9) T_{13}}{T_1 (1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13})} \\ t[11] & 0 & 0 & 0 & 0 \\ t[13] & -\frac{(-1+T_1-T_9)(-1+T_{13})}{1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13}} & 0 & -\frac{(-1+T_1)^2 T_9 (-1+T_{13})}{T_1 (1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13})} & \frac{(-1+T_1)^3 (-1+T_{13})}{T_1 (1-T_1-T_{13}+T_1 T_{13}-T_9 T_{13})} \\ t[15] & 0 & -1+T_{15} & 0 & 0 \\ \sigma & \frac{1}{T_9 T_{13}} & T_{15} & \frac{1}{T_1} & T_1 \end{array} \right) \end{array} \right.$$

$$\{9, \left( \begin{array}{c} \frac{-1+T_1+T_{13}}{T_1 T_{13}} \\ t[1] \\ t[11] \\ t[13] \\ t[15] \\ \sigma \end{array} \right) \left( \begin{array}{ccc} h[1] & h[10] & h[12] \\ -\frac{(-1+T_1)(-1+T_{13}+T_1 T_{13})}{T_1 T_{13}(-1+T_1+T_{13})} & 0 & -\frac{(-1+T_1)(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ 0 & 0 & 0 \\ -\frac{-1+T_{13}}{-1+T_1+T_{13}} & 0 & \frac{(-1+T_1)^2(-1+T_{13})}{-1+T_1+T_{13}} \\ 0 & -1+T_{15} & 0 \\ \frac{1}{T_1 T_{13}} & T_{15} & \frac{1}{T_1} \end{array} \right) \left( \begin{array}{c} h[14] \\ \frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ 0 \\ -\frac{(-1+T_1)^3(-1+T_{13})}{T_1(-1+T_1+T_{13})} \\ 0 \\ T_1 \end{array} \right)$$

$$\{10, \left( \begin{array}{c} \frac{-1+T_1+T_{13}}{T_1 T_{13}} \\ t[1] \\ t[11] \\ t[13] \\ t[15] \\ \sigma \end{array} \right) \left( \begin{array}{ccc} h[1] & h[12] & h[14] \\ -\frac{(-1+T_1)(-1+T_{13}+T_1 T_{13}) T_{15}}{T_1 T_{13}(-1+T_1+T_{13})} & -\frac{(-1+T_1)(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13}) T_{15}}{T_1(-1+T_1+T_{13})} & \frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ 0 & 0 & 0 \\ -\frac{-1+T_{13}}{-1+T_1+T_{13}} & \frac{(-1+T_1)^2(-1+T_{13})}{-1+T_1+T_{13}} & -\frac{(-1+T_1)^3(-1+T_{13})}{T_1(-1+T_1+T_{13})} \\ \frac{T_1(-1+T_{15})}{-1+T_1+T_{13}} & \frac{(-1+T_1)(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13})(-1+T_{15})}{T_1(-1+T_1+T_{13})} & -\frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ \frac{T_{15}}{T_1 T_{13}} & \frac{1}{T_1} & T_1 \end{array} \right) \left( \begin{array}{c} h[14] \\ \frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ 0 \\ -\frac{(-1+T_1)^3(-1+T_{13})}{T_1(-1+T_1+T_{13})} \\ -\frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ T_1 \end{array} \right)$$

$$\{11, \left( \begin{array}{c} \frac{-1+T_1+T_{13}}{T_1 T_{13}} \\ t[1] \\ t[13] \\ t[15] \\ \sigma \end{array} \right) \left( \begin{array}{ccc} h[1] & h[12] & h[14] \\ -\frac{(-1+T_1)(-1+T_{13}+T_1 T_{13}) T_{15}}{T_1 T_{13}(-1+T_1+T_{13})} & -\frac{(-1+T_1)(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13}) T_{15}}{T_1(-1+T_1+T_{13})} & \frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ -\frac{-1+T_{13}}{-1+T_1+T_{13}} & \frac{(-1+T_1)^2(-1+T_{13})}{-1+T_1+T_{13}} & -\frac{(-1+T_1)^3(-1+T_{13})}{T_1(-1+T_1+T_{13})} \\ \frac{T_1(-1+T_{15})}{-1+T_1+T_{13}} & \frac{(-1+T_1)(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13})(-1+T_{15})}{T_1(-1+T_1+T_{13})} & -\frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ \frac{T_{15}}{T_1 T_{13}} & \frac{1}{T_1} & T_1 \end{array} \right) \left( \begin{array}{c} h[14] \\ \frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ -\frac{(-1+T_1)^3(-1+T_{13})}{T_1(-1+T_1+T_{13})} \\ -\frac{(-1+T_1)(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13})}{T_1(-1+T_1+T_{13})} \\ T_1 \end{array} \right)$$

$$\{12, \left( \begin{array}{c} \frac{T_1-T_1^2-T_1 T_{13}+T_{15}-3 T_1 T_{15}+3 T_1^2 T_{15}-T_1^3 T_{15}-T_{13} T_{15}+2 T_1 T_{13} T_{15}-2 T_1^2 T_{13} T_{15}+T_1^3 T_{13} T_{15}}{T_1^2 T_{13}} \\ t[1] \\ t[13] \\ t[15] \\ \sigma \end{array} \right) \left( \begin{array}{c} h[1] \\ \frac{(-1+T_1) T_{15}(-T_1+T_1 T_{13}+T_1^2 T_{13}-T_{15}+2 T_1 T_{13} T_{15})}{T_1^2 T_{13}(T_1-T_1^2-T_1 T_{13}+T_{15}-3 T_1 T_{15}+3 T_1^2 T_{15}-T_1^3 T_{15}-T_{13} T_{15})} \\ \frac{(-1+T_{13})(-T_1-T_{15}+2 T_1 T_{13} T_{15})}{T_1-T_1^2-T_1 T_{13}+T_{15}-3 T_1 T_{15}+3 T_1^2 T_{15}-T_1^3 T_{15}} \\ -\frac{T_1^2(-T_1-T_{15}+2 T_1 T_{13} T_{15})}{T_1-T_1^2-T_1 T_{13}+T_{15}-3 T_1 T_{15}+3 T_1^2 T_{15}-T_1^3 T_{15}} \\ \frac{T_1}{T_1^2} \end{array} \right)$$

$$\{13, \left( \begin{array}{c} \frac{T_1-2 T_1^2+T_{15}-4 T_1 T_{15}+5 T_1^2 T_{15}-3 T_1^3 T_{15}+T_1^4 T_{15}}{T_1^3} \\ t[1] \\ t[15] \\ \sigma \end{array} \right) \left( \begin{array}{c} h[1] \\ -\frac{(-1+T_1)(-T_1^4+T_1 T_{15}-T_1^2 T_{15}-2 T_1^3 T_{15}+2 T_1^4 T_{15}-2 T_1^5 T_{15}+T_1^6 T_{15}+T_{15}^2-3 T_1 T_{15}+2 T_1^2 T_{15}^2-1)}{T_1^3(T_1-2 T_1^2+T_{15}-4 T_1 T_{15}+5 T_1^2 T_{15}-3 T_1^3 T_{15}+T_1^4 T_{15})} \\ -\frac{T_1^2(-1+T_{15})}{T_1-2 T_1^2+T_{15}-4 T_1 T_{15}+5 T_1^2 T_{15}-3 T_1^3 T_{15}+T_1^4 T_{15}} \\ \frac{T_{15}}{T_1^3} \end{array} \right)$$

$$\{14, \left( \begin{array}{c} \frac{1-3 T_1+4 T_1^2-4 T_1^3+T_1^4-T_{15}+4 T_1 T_{15}-7 T_1^2 T_{15}+7 T_1^3 T_{15}-4 T_1^4 T_{15}+T_1^5 T_{15}}{T_1^3} \\ t[1] \\ t[15] \\ \sigma \end{array} \right) \left( \begin{array}{c} h[1] \\ -\frac{(-1+T_1)(-T_1^4+T_{15}-2 T_1 T_{15}+2 T_1^2 T_{15}-3 T_1^3 T_{15}+2 T_1^4 T_{15}-2 T_1^5 T_{15}+T_1^6 T_{15})}{T_1^3(1-3 T_1+4 T_1^2-4 T_1^3+T_1^4-T_{15}+4 T_1 T_{15}-7 T_1^2 T_{15}+7 T_1^3 T_{15}-4 T_1^4 T_{15}+T_1^5 T_{15})} \\ \frac{(1-3 T_1+3 T_1^2-3 T_1^3+T_1^4)(-1+T_{15})}{1-3 T_1+4 T_1^2-4 T_1^3+T_1^4-T_{15}+4 T_1 T_{15}-7 T_1^2 T_{15}+7 T_1^3 T_{15}-4 T_1^4 T_{15}+T_1^5 T_{15}} \\ \frac{T_{15}}{T_1^3} \end{array} \right)$$

$$\{15, \left( \begin{array}{c} \frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^3} \\ t[1] \\ \sigma \end{array} \right) \left( \begin{array}{cc} h[1] & h[16] \\ -\frac{-1+T_1}{T_1} & -1+T_1 \\ \frac{1}{T_1} & T_1 \end{array} \right), \left. h[1] \left(-1 + \frac{1}{T_1}\right) + h[16] (-1 + T_1) \right\}$$



$$\left\{ 16, \left( \begin{array}{cc} -\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^2} & h[1] \\ t[1] & 0 \\ \sigma & 1 \end{array} \right), 0 \right\}$$

$\beta$

$$B \left[ -\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^2}, h[1], 0 \right]$$

`{Alexander[K = Knot[8, 17]][X], Draw[GD[K]]}`

$$\left\{ 11 - \frac{1}{X^3} + \frac{4}{X^2} - \frac{8}{X} - 8X + 4X^2 - X^3, \right. \left. \begin{array}{c} \text{Diagram of a knot with 16 crossings labeled 1 through 16.} \\ \text{Crossings 1-6 are on the top strand, 7-12 on the middle, and 13-16 on the bottom.} \end{array} \right\}$$

`$\beta = \text{Times} @@ \text{GD}[K] /. \{ \text{Ar}[\mathbf{x}_-, \mathbf{y}_-, +1] \Rightarrow \text{R}[\mathbf{x}, \mathbf{y}], \text{Ar}[\mathbf{x}_-, \mathbf{y}_-, -1] \Rightarrow \text{Rinv}[\mathbf{x}, \mathbf{y}] \};$`

`indices = Range[2 Crossings[K]];`

`Table[`

`j = RandomInteger[{1, k - 1}];`

`$\beta = \beta // \text{gm}[\text{indices}[[j]], \text{indices}[[j+1]], \text{indices}[[j]]];$`

`indices = Delete[indices, j + 1];`

`{indices,  $\beta // \beta\text{Form}; \text{Collect}[\text{Last}[\beta] /. t[i_] \Rightarrow 1, \_h, \text{Simplify}[1+\#] \&],$`

`{k, 2 Crossings[K], 2, -1}`

`] // ColumnForm`

$$\begin{aligned} & \{ \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}, h[6] T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_5} + h[14] T_7 + \frac{h[3]}{T_9} + h[16] \\ & \{ \{1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16\}, h[6] T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_5} + h[14] T_7 + \frac{h[3]}{T_8} + h[16] T_1 \\ & \{ \{1, 2, 3, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16\}, h[6] T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_5} + h[14] T_6 + \frac{h[3]}{T_8} + h[16] T_{11} + \\ & \{ \{1, 2, 3, 6, 8, 10, 11, 12, 13, 14, 15, 16\}, h[6] T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_3} + h[14] T_6 + \frac{h[3]}{T_8} + h[16] T_{11} + \frac{h[2]}{T_{13}} \\ & \{ \{1, 2, 3, 6, 8, 10, 11, 12, 13, 14, 16\}, h[6] T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_3} + h[14] T_6 + \frac{h[3]}{T_8} + h[16] T_{11} + \frac{h[2]}{T_{13}} + h \\ & \{ \{1, 2, 3, 8, 10, 11, 12, 13, 14, 16\}, \frac{h[8]}{T_3} + \frac{h[12]}{T_3} + h[14] T_3 + \frac{h[3]}{T_8} T_1 + h[16] T_{11} + \frac{h[2]}{T_{13}} + h[10] T_{14} \} \\ & \{ \{1, 2, 3, 8, 10, 11, 12, 13, 14\}, \frac{h[8]}{T_3} + \frac{h[12]}{T_3} + \frac{h[3]}{T_8} T_1 + h[14] T_3 T_{11} + \frac{h[2]}{T_{13}} + h[10] T_{14} \} \\ & \{ \{1, 2, 8, 10, 11, 12, 13, 14\}, \frac{h[8]}{T_2} + \frac{h[12]}{T_2} + h[14] T_2 T_{11} + \frac{h[2]}{T_8 T_{13}} T_1 + h[10] T_{14} \} \\ & \{ \{1, 2, 8, 11, 12, 13, 14\}, \frac{h[12]}{T_2} + h[14] T_2 T_{11} + \frac{h[2]}{T_8 T_{13}} T_1 + \frac{h[8]}{T_2} T_{14} \} \\ & \{ \{1, 2, 8, 11, 13, 14\}, \frac{h[11]}{T_2} + h[14] T_2 T_{11} + \frac{h[2]}{T_8 T_{13}} T_1 + \frac{h[8]}{T_2} T_{14} \} \\ & \{ \{1, 2, 11, 13, 14\}, \frac{h[11]}{T_2} + h[14] T_2 T_{11} + \frac{h[2]}{T_2^2 T_{13}} T_1 T_{14} \} \\ & \{ \{1, 2, 11, 14\}, \frac{h[11]}{T_2} + h[14] T_2 T_{11} + \frac{h[2]}{T_2^2 T_{11}} T_1 T_{14} \} \\ & \{ \{1, 2, 11\}, \frac{h[2]}{T_2^2} T_1 + h[11] T_{11} \} \\ & \{ \{1, 11\}, \frac{h[1]}{T_1} + h[11] T_{11} \} \\ & \{ \{1\}, 1 \} \end{aligned}$$

`$\beta // \beta\text{Form}$`

$$\left( \begin{array}{cc} -\frac{1-4 T_1+8 T_1^2-11 T_1^3+8 T_1^4-4 T_1^5+T_1^6}{T_1^2} & h[1] \\ t[1] & 0 \\ \sigma & 1 \end{array} \right)$$